

PINBALL DYNAMICS: UNLIMITED ENERGY GROWTH IN NONSMOOTH/SWITCHING HAMILTONIAN SYSTEMS

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ABSTRACT. A family of discontinuous symplectic maps on the cylinder is considered. This family arises naturally in nonsmooth Hamiltonian dynamics and in switched Hamiltonian systems. The transformation depends on two parameters and is a canonical model for the study of bounded and unbounded behavior in discontinuous mappings due to nonlinear resonances. This paper provides a general description of the map and presents one case of an unbounded orbit.

1. INTRODUCTION.

Theory of small perturbations of completely integrable Hamiltonian systems has a long history that goes back to 19th century effort to explain stability of planets. The major breakthrough occurred in the late 1960s, when Kolmogorov-Arnold-Moser (KAM) theory was created.

The KAM theory states that under some non-degeneracy conditions, stable motion persists in a completely integrable Hamiltonian system under sufficiently small and smooth perturbation.

For the original application, 3-body problem, the smoothness was not an issue as the gravity force is analytic, outside of a small set of singularities. However, further applications of KAM theory to stability problems in physics and engineering, do require limited smoothness assumptions and also weaker forms of the so-called twist (nonlinearity) conditions.

The degree of smoothness of the perturbation has a crucial role in the theory. In his famous ICM lecture Kolmogorov gave an outline of the theory where he required analyticity. Shortly, V.I. Arnold proved Kolmogorov's statement, also under the assumption of analyticity. Independently, combining Kolmogorov's method with Nash smoothing technique, Moser proved a KAM type theorem requiring 333 derivatives. Subsequently, the smoothness requirement was reduced to single digits (C^3) and several counterexamples have been found for lower regularity maps, see *e.g.* [5].

Moser proved his theorem for the case of area-preserving monotone twists maps of the annulus. In this article we also restrict our attention to the representative case of twist maps on the plane, which corresponds to the periodically forced Hamiltonian systems with one degree of freedom.

The above KAM counterexamples, that were constructed for the general twist maps, do not provide a tool to decide stability in specific physics problems. Therefore, it is important to investigate special maps arising in physics and geometry.

We note that even in the most extreme case of discontinuous maps, the stability problem is already nontrivial. In the next section, we review several such systems where boundedness problem for discontinuous maps naturally arises. Then we introduce a family of discontinuous twist maps, which captures the essential properties of those examples. The family contains a natural physical system which we call pinball transformation. The hallmark of the pinball map is the small twist, which on the one hand frequently occurs in applications, and on the other hand makes stability problem rather delicate.

2. DISCONTINUOUS TWIST MAPS AND $\alpha\mathbf{z}$ -TRANSFORMATION.

Discontinuous maps arise naturally in Hamiltonian systems with impacts, such as Fermi-Ulam problem, billiards, and more recently in hybrid or switched systems. It is worth noticing that in each of these examples, under the additional smoothness assumptions, KAM theory applies assuring boundedness of energy.

One should also keep in mind that while the general monotone twist maps are characterized by a function of two variables $h(x_1, x_2)$, these particular examples correspond to symplectic maps characterized by function of one variable, *e.g.* for billiards $h(x_1, x_2) = \|x_2 - x_1\|$. Such a restriction makes it nontrivial to construct physically meaningful escaping trajectories.

For the readers' convenience, now we briefly describe these systems.

Example 1: Particle in square wave switching potential. Hybrid or switching systems is an active area of research in applied mathematics and engineering sciences, see *e.g.* [9],[1]. A prototype example of a switching system, where boundedness problem is already non-trivial, is a classical particle in square wave periodic potential which changes the sign, periodically in time.

More precisely, let the potential be $V(x) = (-1)^{[x]}$ and assume the potential is switched every second $V(x, t) = (-1)^{[t]} \cdot (-1)^{[x]}$. While such potential is not differentiable, there is a natural way to define the dynamics by using the energy relation: the kinetic energy changes by 2 if the particle passes $t \in \mathbb{Z}$ integer points. It is common to ignore the singular subset of the extended phase space (t, \dot{x}, x) where there is discontinuity in both time and space and the dynamics is not defined. Such subset has zero measure. Outside the singular set particle moves with constant speed $v = \sqrt{E \pm 1}$, gaining or losing energy by one at each switching. See the appendix for more details.

Example 2: Fermi-Ulam accelerator. The Fermi-Ulam system consists of a classical particle bouncing between two periodically moving walls. The application of KAM theory shows that velocity (or energy) of the particle is uniformly bounded $|\dot{v}| < C(v(0))$, provided the periodically moving wall's position is sufficiently smooth $p(t) \in C^5(0, T)$ [7].

Fermi-Ulam problem can be reduced to a particle traveling in a periodic non-smooth potential

$$\ddot{x} + V'(x, t) = 0, \text{ where } V' \in L^1(\mathbb{T}^2).$$

It turns out that lack of smoothness in x (*e.g.* due to the presence of the wall in Fermi-Ulam problem) does not destroy bounded behavior as one can exchange the role of time and coordinate and then obtain a smooth monotone twist map by integrating over x , see *e.g.* [8].

If there is lack of smoothness in both space and time in the periodic potential problem, then KAM theorems do not apply.

In the worst case the map is discontinuous, but even then, finding unbounded solutions could be challenging. One case, however, is more tractable: if jumps in the velocity are so large that the solution makes full revolution over one period of forcing so it will be in tune for the next velocity increase. A typical example would be given by this map

$$(1) \quad \begin{cases} x_1 = (x + y) \mod 1 \\ y_1 = y + \operatorname{sgn}(x_1 - \frac{1}{2}). \end{cases}$$

Such scenario takes place in Fermi-Ulam problem if $p(t)$ has saw-tooth like shape. But, if the velocity increments are smaller, then the twist will eventually detune the solution out of resonance.

Example 3. Outer Billiards. The question of boundedness becomes a lot more delicate and there are few examples of escaping trajectories for such systems in the dual billiards. Only recently, Schwartz and then Dolgopyat and Fayad constructed unbounded solutions in the presence of

piecewise smooth boundary. In the appendix, we give some heuristic description how the discontinuous twist map arises in this example.

αz -Map: A model of boundedness problem for discontinuous twist maps.

In this paper, we introduce a two-parameter family of discontinuous monotone twist maps that seems to capture the essential difficulties of several switching-like (discontinuous) systems. The map is given by

$$(2) \quad \begin{cases} x_1 = (x + \alpha y^z) \mod 1 \\ y_1 = y + \text{sgn}(x_1 - \frac{1}{2}) \end{cases}$$

and will be referred to as αz -map. Notice, that the map is invariant with respect to the scaling that varying the amplitude of the changes in the second variable or varying the length of the base circle in the first variable will lead to the equivalent system with different values of parameter α . We also observe that αz transformation preserves the unit-step lattice in action variable. In other words, the action variable is quantized for any fixed initial condition.

For different values of parameters αz map corresponds to some natural systems:

- $z = 1$, Fermi-Ulam with saw-tooth $p(t)$, discontinuous standard map.
- $z = 0$, Kesten system (skew product of irrational rotation with jumps). It is defined in the next section.
- $z = 1/2$, Particle in switching square wave periodic potential
- $z = -1$, Pinball problem (studied in this paper)

We explain in more details how αz -transformation arises in each of these examples in the appendix.

Zero twist example. Kesten system. The following system was introduced by Kesten in [6] independently of any connection with KAM theory-type of problems. He considered irrational rotation on the circle and asked what is the discrepancy between the orbit visiting different open subsets of the circle having equal measure. In particular one can consider different halves of the circle $x \in (0, 1/2)$ and $x \in (1/2, 1)$. In our notation, his system corresponds to the map with $z = 0$.

In this degenerate case, there is no twist in the system and the dynamics is a skew product. Thus, one can easily provide a set of values of parameter α (e.g. $\alpha = 1$) for which there are unbounded orbits. On the other hand for $\alpha = \frac{1}{2}$ any trajectory of the system (2) is bounded since any point has period exactly 2. In the generic case of irrational values of α , Kesten's question leads to an interesting number-theoretic problem. Some results can be found in his original paper [6] where it is stated that for almost every α positive measure set of orbits escapes to infinity but return to zero infinitely often. Most contemporary analysis of this phenomena can be found in [11].

Surprisingly, Kesten's system becomes important in the study of discontinuous twist maps after an appropriate renormalization procedure is carried out.

Elementary properties of αz -map. For non-degenerate twist $z \neq 0$ the following properties hold:

- For any $z < -1$ nearly half of trajectories of the system (2) escapes to infinity. It immediately follows from the fact that $\sum_n n^z$ converges.
- Fix $z \in \mathbb{N}$ then for $\alpha = 1$, it is easy to verify that half the orbits are unbounded and for $\alpha = \frac{1}{2}$ all the trajectories are periodic.
- The most interesting and difficult problem of boundedness occurs for $z \in [-1, 1)$.

3. PINBALL SYSTEM

Consider now Fermi-Ulam like system with the fixed walls, but with one of the walls containing a pinball mechanism: the momentum of the particle increases or decreases when it hits the wall according to the following law:

$$(3) \quad \begin{cases} v \rightarrow v + 1 & \text{if } t \in [0, 1/2] \pmod{1} \\ v \rightarrow v - 1 & \text{if } t \in [1/2, 1] \pmod{1}, \end{cases}$$

i.e. the momentum is increased (decreased) during the first (second) half period. This dynamics is described by the map with α being the distance between the walls and $z = -1$.

We rewrite the system (2) for $z = -1$ and we will call such system a pinball transformation that will be denoted by \mathbf{P} . For the sake of clarity, it would be more convenient to consider the base circle $\varphi \in [0, 2)$.

$$(4) \quad \begin{cases} \varphi_1 = \left(\varphi + \frac{\alpha}{I} \right) \pmod{2} \\ I_1 = I + \text{sgn}(\varphi_1 - 1) \end{cases}$$

Numerical experiments show that for typical values of parameter, the trajectory of the system (4) is nearly recurrent for a long time and moreover approximates some piecewise smooth function having singularities only at the discontinuity lines $\varphi = 1$ and $\varphi = 0$.

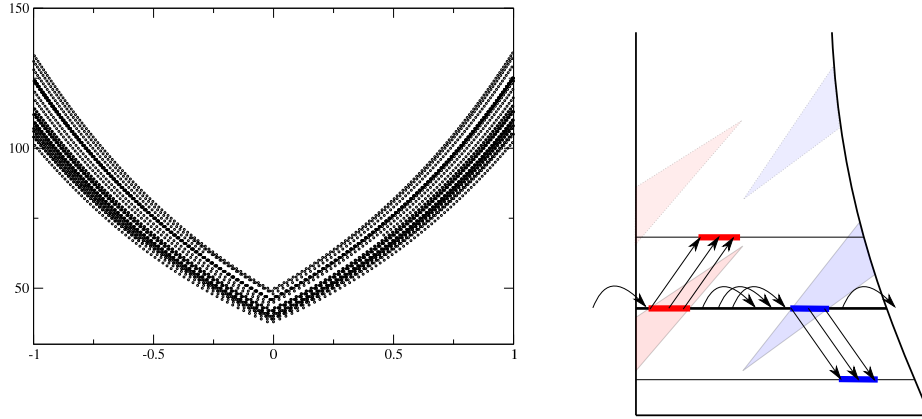


FIGURE 1. Left: typical trajectory in a phase space: 10^6 iterations starting at $I = 50$. Right: Fundamental domain Φ .

Our goal is to explain this behavior by renormalizing the induced transformation in the so-called fundamental domain Φ . The fundamental domain is related to Poincaré' section for flows in the sense that any orbit returns to it. In the Pinball map, we define the fundamental domain as the set of points (φ, I) laying between singular line $\varphi = 0$ and its image

$$(5) \quad \Phi = \{(\varphi, I) \in [0, 2) \times [0, \infty], 0 < \varphi < \text{Proj}_\varphi \mathbf{P}(0, I)\}.$$

Our first statement concerns the asymptotic description of the first return map:

THEOREM 1. *First return map of the domain Φ under the map (4) is a $O(I^{-2})$ perturbation of the Kesten transformation (2) with rotation number $\mu = \exp \alpha^{-1}$, provided $\alpha > 1/\ln 3$.*

If μ is rational, than the reduced Kesten system might possesses a uniformly growing trajectory. Indeed, if one of the periodic points stays longer in the positive part of the base interval

than in the negative part, then the corresponding trajectory grows without bound. Our construction of an escaping trajectory of the system (4) consists in choosing the initial data in an appropriate way so as to kill the leading order perturbation of Kesten's map. Next, we would have to estimate that the remaining perturbation will not destroy such "resonant" growth. Combining these ideas, we prove

THEOREM 2. *For $\alpha = \frac{1}{\ln 2}$ there exists an unbounded trajectory in the system (4).*

Note that smoothing the signum function discontinuity in (4) will make KAM theory applicable and then all solutions will be bounded.

Indeed, change the variables: $(\varphi, y) = (\varphi, I^{-1})$. In the new variables, the smooth version of the Pinball transformation takes the form

$$\begin{cases} \varphi_1 = \varphi + \alpha y \mod 2 \\ y_1 = \frac{1}{\frac{1}{y} - f(\varphi, y)} \end{cases}$$

where f is smooth and $|f(\varphi, y)| < 1$. Then

$$y_1 = y \frac{1}{1 - yf(\varphi, y)} = y \left(1 + \sum_{n=1}^{\infty} (yf(\varphi, y))^n \right)$$

so the perturbation is of order $O(y^2)$ which is much smaller than the twist. The curve intersection property follows from the area-conservation in the original variables. Therefore, this map satisfies the conditions of monotone twist theorem, see *e.g.* [10].

4. PROOF OF THEOREM 1.

Define a subset $\Phi \subset \mathbb{S} \times \mathbb{R}_+$ between the discontinuity line $\phi = 0$ and its first iteration:

$$(6) \quad \Phi = \left\{ (\varphi, I) \mid \varphi \in \left(0, \frac{\alpha}{I-1} \right) \right\}$$

and consider the transformation $T(\varphi, I) = (\varphi', I')$ as the first return map for any point $(\varphi, I) \in \Phi$ according to (4). We have the following bound on the action change

LEMMA 1. *If (φ', I') is an image of the point (φ, I) under the transformation T then $|I' - I| \leq 1$.*

In other words, Lemma 1 states that as the angle variable winds around the cylinder and the action variable undergoes large changes, after returning to the fundamental domain, the action will not change by more than 1. This statement gives a good local control on the orbits.

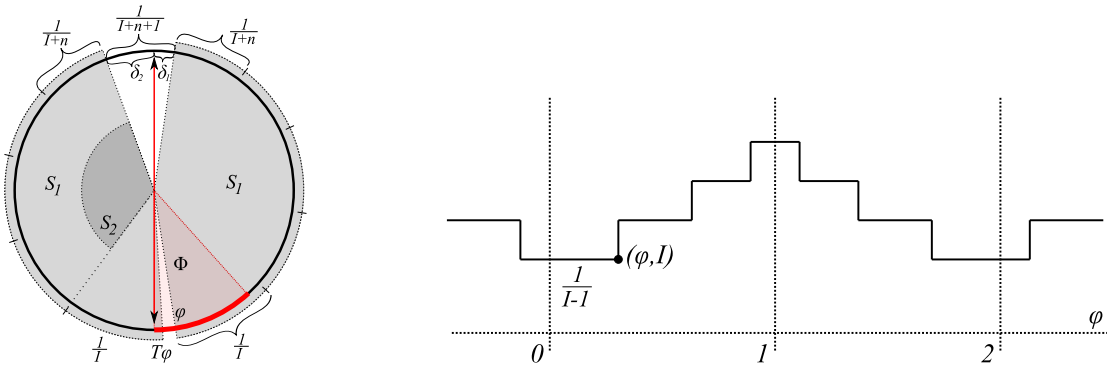


FIGURE 2. Mapping from fundamental domain to itself represented on the base circle (right panel) and on the covering space (left panel).

Next we describe the structure of the subsets in the base $\varphi \in (0, \alpha/(I-1))$ for which the action increases $I' = I + 1$ or decreases $I' = I - 1$.

Let us introduce the *rescaled* angle variable $\tilde{\varphi} = (I-1)\varphi\alpha^{-1}$. In the renormalized variables, the fundamental domain can be represented by

$$\Phi = (\tilde{\varphi}, I) = [0, 1] \times \mathbb{R}_+.$$

LEMMA 2. *The set $\Phi \cap \{I = C\}$ is the union of three subsets*

$$\Phi \cap \{I = C\} = \mathcal{I}_+ \cup \mathcal{I}_- \cup \mathcal{I}_0,$$

where \mathcal{I}_+ and \mathcal{I}_- are intervals of equal measure and consist of all points (φ, C) such that $T(\varphi, C) = (\cdot, C \pm 1)$ respectively.

The intervals are contained in the regions

$$\mathcal{I}_+ \in \left(0, \frac{\alpha}{\mu(I-1)}\right) \quad \text{and} \quad \mathcal{I}_- \in \left(\frac{\alpha(1-\mu^{-1})}{I-1}, \frac{\alpha}{I-1}\right),$$

where $\mu(\alpha) = \exp(\alpha^{-1})$.

Finally, we have the following lemma which ends the proof of Theorem 1.

LEMMA 3. *If $\alpha > 1/\ln 3$, then the first return map T in the rescaled variables takes the form*

$$T(\varphi, I) = \begin{cases} \tilde{\varphi}_1 = \tilde{\varphi} + 2/\mu(\alpha) + O(I^{-1}) \\ I_1 = I + \chi_+(\tilde{\varphi}_1) - \chi_-(\tilde{\varphi}_1) \end{cases}$$

where $\mu(\alpha) = \exp(\alpha^{-1})$, χ_{\pm} are characteristic functions of positive and negative intervals depending weakly on I .

5. PROOFS

5.1. Proof of Lemma 1.

Proof. We begin with giving a heuristic argument based on the Figure 2 (right panel). We can represent orbits as stairs going up and down in the domain $\varphi \in [0, 2]$. Consider the special configuration that is symmetric with respect to $\varphi = 1$. It is easy to see that the corresponding solution will have the same action after the first return map. By moving the graph so that the top level does not cross $\varphi = 1$ it is possible to see that the change in the action cannot be more than one. Indeed, the lowest level steps are wider than the top one and therefore at most one crossing can occur.

Now, we provide the full proof. Assume the initial point (φ, I) is in the fundamental domain Φ , i.e. $\varphi \in (0, \alpha/(I-1))$. The map (4) is iterated $n+1$ times while I increases until the orbit is one step away from crossing $\varphi = 1$. Next, the map is iterated $n'+1$ times while I decreases until the orbit is one step away from crossing $\varphi = 2$. Then, by the definition of the fundamental domain, last $(n+n'+2)$ -nd iterate is in Φ and we obtain

$$(7) \quad T(\varphi, I) = \left(\varphi + \alpha S_1(\varphi, I) + \alpha S_2(\varphi, I) + \frac{\alpha}{I+n+1} + \frac{\alpha}{I+(n-n'-1)}, I+n-n' \right),$$

where

$$(8) \quad S_1(\varphi, I) = \sum_{k=0}^n \frac{1}{I+k}, \quad S_2(\varphi, I) = \sum_{k=0}^{n'} \frac{1}{I+n-k}.$$

The numbers n and n' are uniquely defined by the relations:

$$(9) \quad \begin{cases} 1 > \varphi + \alpha S_1(\varphi, I) \\ 1 < \varphi + \alpha S_1(\varphi, I) + \frac{\alpha}{I + n + 1} \end{cases}$$

and

$$(10) \quad \begin{cases} 2 > \varphi + \alpha S_1(\varphi, I) + \alpha S_2(\varphi, I) + \frac{\alpha}{I + n + 1} \\ 2 < \varphi + \alpha S_1(\varphi, I) + \alpha S_2(\varphi, I) + \frac{\alpha}{I + n + 1} + \frac{\alpha}{I + n - n' - 1}. \end{cases}$$

The equation (9) means that n -th iterate is the last one staying in the right half-circle, so the next iterate will be in the left half-circle. Similarly $(n + 1) + n'$ -th iterate is the last one before returning to Φ , see Figure 2.

Denote by $\Delta S = S_2 - S_1$. Rewriting the second sum in (8) for $k' = n - k$ we obtain for ΔS :

$$(11) \quad \Delta S = \sum_{k'=n-n'}^n \frac{1}{I + k'} - \sum_{k=0}^n \frac{1}{I + k}.$$

Note that the expression (11) implies that

$$(12) \quad \begin{cases} \Delta S > \frac{1}{I-1} + \frac{1}{I-2} & \text{if } n - n' < -1 \\ \Delta S < -\frac{1}{I} - \frac{1}{I+1} & \text{if } n - n' > 1. \end{cases}$$

To prove that $|n - n'| \leq 1$ it is sufficient to show that (12) contradicts (9) – (10). Rewriting (10) for $S_2 = \Delta S + S_1$ we obtain

$$(13) \quad 2 > \varphi + 2\alpha S_1(\varphi, I) + \alpha \Delta S + \frac{\alpha}{I + n + 1} > 2 - \frac{\alpha}{I + n - n' - 1}.$$

Multiplying the equation (9) by 2 and substituting into (13) we get

$$(14) \quad \begin{cases} \Delta S < \frac{\varphi}{\alpha} + \frac{1}{I + n + 1} \\ \Delta S > \frac{\varphi}{\alpha} - \frac{1}{I + n + 1} - \frac{1}{I + n - n' - 1}. \end{cases}$$

Since for any point $(\varphi, I) \in \Phi$ one has $\frac{\varphi}{\alpha} \in \left(0, \frac{1}{I-1}\right)$ inequalities (14) contradict (12). □

5.2. Proof of lemma 2.

Proof. As already found in the proof of Lemma 1, transformation $T : \Phi \rightarrow \Phi$ in φ -variable takes the form

$$(15) \quad \varphi' = \varphi + \left(2\alpha S_1(\varphi, I) + \frac{\alpha}{I + n + 1} - 2\right) + \alpha \Delta S,$$

where

$$(16) \quad \Delta S = \begin{cases} 0, & \text{if } \varphi \in \mathcal{I}_+ \\ \frac{1}{I-1}, & \text{if } \varphi \in \mathcal{I}_0 \\ \frac{1}{I-1} + \frac{1}{I-2}, & \text{if } \varphi \in \mathcal{I}_- \end{cases}$$

Define δ_1, δ_2 by the relations (see Fig. 2)

$$\begin{aligned}\varphi + \alpha S_1(\varphi, I) &= 1 - \delta_1 \\ \varphi + \alpha S_1(\varphi, I) + \frac{\alpha}{I + n + 1} &= 1 + \delta_2,\end{aligned}$$

then it is easy to see that

$$\varphi + 2\alpha S_1(\varphi, I) + \frac{\alpha}{I + n + 1} - 2 = \delta_2 - \delta_1 - \varphi.$$

Then the expression (16) could be rewritten in the form

$$(17) \quad \begin{cases} \delta_2 - \delta_1 - \varphi > 0, & \text{for } \varphi \in \mathcal{I}_+ \\ \delta_2 - \delta_1 - \varphi \in \left(-\frac{\alpha}{I-1}, 0\right), & \text{for } \varphi \in \mathcal{I}_0 \\ \delta_2 - \delta_1 - \varphi < -\frac{\alpha}{I-1}, & \text{for } \varphi \in \mathcal{I}_- \end{cases}$$

In particular, the first inequality in (17) implies that for $\varphi \in \mathcal{I}_+$ one has

$$\varphi < \frac{\alpha}{I + n + 1}, \text{ since } \frac{\alpha}{I + n + 1} = \delta_2 + \delta_1.$$

Similarly, it is easy to verify that if $\varphi \in \mathcal{I}_-$ then $\varphi > \frac{\alpha}{I-1} - \frac{\alpha}{I+n+1}$. The next proposition describes some rigidity properties of the intervals $\mathcal{I}_0, \mathcal{I}_+, \mathcal{I}_-$ in (17)

PROPOSITION 1. *Assume that $(\varphi_0, I) \in \mathcal{I}_+$ and that the corresponding sum $S_1(\varphi_0, I)$ has n terms. Then the point $(\varphi_\epsilon = \varphi_0 + \epsilon, I)$ belongs to \mathcal{I}_+ if and only if the sum $S_1(\varphi_\epsilon, I)$ has also n terms.*

Proof. Denote by $\delta_1^{(0)}$ and $\delta_2^{(0)}$ the corresponding parts of the interval $\frac{\alpha}{I+n+1}$ for the point φ_0 . By assumption $\varphi_0 \in \mathcal{I}_+$ and thus $\delta_2^{(0)} - \delta_1^{(0)} - \varphi_0 > 0$, then by (17) it is sufficient to verify that $\delta_2^{(\epsilon)} - \delta_1^{(\epsilon)} - \varphi_\epsilon$ remains positive.

Then the proof can be obtained from the following calculation:

(a) First, consider the case when the number of terms remains the same (equal to n). Then $\delta_2^{(\epsilon)} = \delta_2 + \epsilon$, $\delta_1^{(\epsilon)} = \delta_1 - \epsilon$ and so

$$(18) \quad \delta_2^{(\epsilon)} - \delta_1^{(\epsilon)} - \varphi = (\delta_2^{(0)} + \epsilon) - (\delta_1^{(0)} - \epsilon) - \epsilon = \delta_2^{(0)} - \delta_1^{(0)} + \epsilon > \delta_2^{(0)} - \delta_1^{(0)}$$

(b) Assume now that the number of terms in $S_1(\varphi_\epsilon, I)$ is different from n . Assume it contains $n-1$ terms (all other cases can be treated similarly). Then we have, see Figure 3.

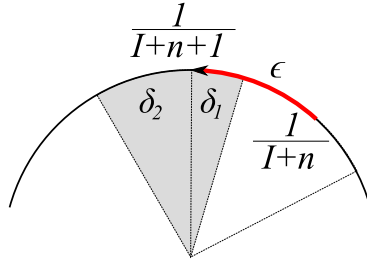


FIGURE 3. Rotation by ϵ .

$$\delta_2^{(\epsilon)} = \epsilon - \delta_1^{(0)} = \epsilon + \delta_2^{(0)} - \frac{\alpha}{I + n + 1}$$

which implies

$$\delta_1^{(\epsilon)} = \frac{\alpha}{I + n} - \delta_2^{(\epsilon)} = \frac{\alpha}{I + n} - \epsilon + \delta_1^{(0)}.$$

Finally, we have

$$\delta_2^{(\epsilon)} - \delta_1^{(\epsilon)} - \varphi_\epsilon = \epsilon + \delta_2^{(0)} - \delta_1^{(0)} - \varphi_0 - \frac{\alpha}{I + n + 1} - \frac{\alpha}{I + n}.$$

The latter expression is negative since by construction

$$\delta_2^{(0)} + \epsilon < \frac{1}{I + n} + \frac{1}{I + n + 1}.$$

This ends the proof of the proposition. \square

Using proposition 1, we will find the set \mathcal{I}_+ . Let the initial angle be $\varphi_0 = 0$ then

$$\delta_2^{(0)} = \left\{ \alpha \sum_{j=0}^{n+1} \frac{1}{I + j} \right\} \text{ and } \delta_1^{(0)} = \frac{\alpha}{I + n + 1} - \delta_2^{(0)}.$$

If $\delta_2^{(0)} > \delta_1^{(0)}$ then by (17) the point $(0, I)$ belongs to \mathcal{I}_+ and then proposition 1 implies that $\mathcal{I}_+ = (0, \delta_1^{(0)})$.

Otherwise, if $\delta_2^{(0)} < \delta_1^{(0)}$ then $(0, I)$ does not belong to \mathcal{I}_+ and to find the leftmost point $(\phi = \epsilon, I) \in \mathcal{I}_+$ we need to satisfy the condition $\delta_2^{(\epsilon)} - \delta_1^{(\epsilon)} - \epsilon > 0$. Using (18) we obtain $\epsilon = \delta_1^{(0)} - \delta_2^{(0)}$ and therefore in this case $\mathcal{I}_+ = (\delta_1^{(0)} - \delta_2^{(0)}, \delta_1^{(0)})$.

Similar calculations can be carried out for \mathcal{I}_- , but it is easier to use symmetric properties of the map with respect to reversing the “time”.

Then, if $\delta_1^{(00)} > \delta_2^{(00)}$ ($\delta_i^{(00)}$ correspond to $\delta^{(0)}$ but are obtained from negative iterates) the point $(0-, I - 1)$ which is the preimage of $(\frac{\alpha}{I-1}, I)$ belongs to \mathcal{I}_- and by the same argument as above for \mathcal{I}_+ , we have

$$\mathcal{I}_- = \left(\frac{\alpha}{I - 1} - \delta_2^{(00)}, \frac{\alpha}{I - 1} \right).$$

If the converse holds, *i.e.* $\delta_2^{(00)} > \delta_1^{(00)}$ we obtain

$$\mathcal{I}_- = \left(\frac{\alpha}{I - 1} - (\delta_2^{(00)} - \delta_1^{(00)}), \frac{\alpha}{I - 1} - \delta_2^{(00)} \right).$$

This ends the proof of the lemma. \square

The location and the measure of the intervals \mathcal{I}_+ and \mathcal{I}_- are then controlled by the fractional parts of the sums $\alpha S_1(0, I)$. We shall estimate these quantities in the proof of the next lemma.

5.3. Proof of Lemma 3.

Proof. For the clarity of presentation, we give the proof only for the case $I = N$. The general case can be treated similarly. We introduce the notation for the expression

$$\alpha\psi(\varphi, I) = 2\alpha S_1(\varphi, I) + \frac{\alpha}{I + n + 1} - 2$$

from (15).

In the rescaled variables, the transformation (15) takes the form

$$(19) \quad \tilde{\varphi}' = \begin{cases} \frac{I}{\alpha} \varphi' = \left(1 + \frac{1}{I-1}\right) \tilde{\varphi} + I\psi(I, \varphi), & \varphi \in \mathcal{I}_+ \\ \frac{I-1}{\alpha} \varphi' = \tilde{\varphi} + (I-1)\psi(I, \varphi) + 1, & \varphi \in \mathcal{I}_0 \\ \frac{I-2}{\alpha} \varphi' = \left(1 - \frac{1}{I-1}\right) \tilde{\varphi} + (I-2)\psi(I, \varphi) + 2 - \frac{1}{I-1}, & \varphi \in \mathcal{I}_- \end{cases}$$

Case $\varphi \in \mathcal{I}_+$. If the action variable I increases, then the number of iterates of the transformation (4) contained in the right half-circle (where the action is gained) is greater than the number of iterates contained in the left half-circle. Lemma 1 implies that $n = n' + 1$. Thus, the transformation T for these values of φ takes the form

$$\varphi' = \varphi + 2\alpha S_1(\varphi, I) + \frac{\alpha}{I + n + 1} - 2,$$

where

$$S_1(\varphi, N) = \sum_{k=0}^n \frac{1}{N+k} = H_{N+n} - H_{N-1},$$

where H_k denotes k -th harmonic number. The harmonic number has the following asymptotic expansion

$$H_k = \ln k + \gamma + \frac{1}{2k} - \sum_{j=1}^{\infty} \frac{B_{2j}}{2^j k^{2j}}$$

where B_j denotes j -th Bernoulli number. Thus,

$$(20) \quad \begin{aligned} S_1(\varphi, N) = & \ln \frac{N+n}{N-1} + \frac{1}{2} \left(\frac{1}{N+n} - \frac{1}{N-1} \right) - \\ & - \frac{1}{12} \left(\frac{1}{(N+n)^2} - \frac{1}{(N-1)^2} \right) + O\left(\frac{1}{N^4}\right) \end{aligned}$$

We will now show that $n = [\mu(N-1)] - N$, where $\mu = \exp(1/\alpha)$. By proposition 1, the number of steps in the positive half of the cylinder n is independent of $\varphi \in \mathcal{I}_+$, therefore it is sufficient to verify $n = [\mu(N-1)] - N$ only for some φ using (9). Moreover, we can take $\varphi = 0$ even though this point might not be in \mathcal{I}_+ . Indeed, according to lemma 2 the point $\varphi = 0$ satisfies $\varphi \in \mathcal{I}_+$ or $\varphi \in \mathcal{I}_0$. If the former, we are done. If the latter then it is easy to see that by increasing ϕ we will eventually cross into \mathcal{I}_+ without changing n : the number of steps in positive part of the cylinder.

We have

$$\begin{aligned} S_1(\varphi, N) = & \ln \frac{[\mu(N-1)]}{N-1} + \frac{1}{2(N-1)} \left(\frac{N-1}{[\mu(N-1)]} - 1 \right) + \\ & + \frac{1}{12(N-1)^2} \left(\frac{(N-1)^2}{[\mu(N-1)]^2} - 1 \right) + O\left(\frac{1}{N^4}\right). \end{aligned}$$

Using the relation $\mu(N-1) = [\mu(N-1)] + \{\mu(N-1)\}$ and denoting $y := \frac{\{\mu(N-1)\}}{\mu(N-1)}$ rewrite

$$\begin{aligned} S_1(\varphi, N) = & \ln \mu(1-y) + \frac{1}{2(N-1)} \left(\frac{1}{\mu(1-y)} - 1 \right) - \\ & - \frac{1}{12(N-1)^2} \left(\frac{1}{\mu^2(1-y)^2} - 1 \right) + O\left(\frac{1}{N^4}\right). \end{aligned}$$

Expanding in perturbation series for y we obtain

$$\begin{aligned}
(21) \quad S_1(\varphi, N) &= \ln \mu + \frac{1}{2(N-1)} \left(\frac{1}{\mu} - 1 \right) - \frac{1}{12(N-1)^2} \left(\frac{1}{\mu^2} - 1 \right) + \\
&+ \sum_{k=1}^{\infty} y^k \left(\frac{1}{2\mu(N-1)} - \frac{1}{12\mu^2(N-1)^2} - \frac{1}{k} \right) + O\left(\frac{1}{N^4}\right).
\end{aligned}$$

Recall that $\alpha = \frac{1}{\ln \mu}$ and $1 < \mu < 3$. Then using (21) it easy to verify that $\alpha S_1(\varphi, N) < 1$ in the limit $N \rightarrow \infty$ as the leading order terms containing $\frac{1}{N-1}$ are negative. On the other hand, $\frac{1}{N+n+1} = \frac{1}{[\mu(N-1)]+1}$ and by similar estimates $\alpha S_1(\varphi, N) + \frac{\alpha}{N+n+1} > 1$. This shows that $n = [\mu(N-1)] - N$ is the correct number of steps.

Now, we will derive an explicit expression for the first return map T . Multiply (21) by 2α

$$\begin{aligned}
(22) \quad 2\alpha S_1(\varphi, N) &= 2 + \frac{\alpha}{(N-1)} \left(\frac{1}{\mu} - 1 \right) - \frac{\alpha}{6(N-1)^2} \left(\frac{1}{\mu^2} - 1 \right) + \\
&+ \alpha \sum_{k=1}^{\infty} y^k \left(\frac{1}{\mu(N-1)} - \frac{1}{6\mu^2(N-1)^2} - \frac{1}{k} \right) + O\left(\frac{1}{N^4}\right)
\end{aligned}$$

so the transformation T takes the form

$$\begin{aligned}
(23) \quad \varphi' &= \left(\varphi + 2\alpha S_1(\varphi, N) + \frac{\alpha}{[\mu(N-1)]+1} \right) \mod 2 = \\
&= \left(\varphi + \frac{\alpha}{(N-1)} \left(\frac{1}{\mu} - 1 \right) - \frac{\alpha}{6(N-1)^2} \left(\frac{1}{\mu^2} - 1 \right) + \frac{\alpha}{1 + [\mu(N-1)]} + \right. \\
&\left. + \alpha \sum_{k=1}^{\infty} y^k \left(\frac{1}{\mu(N-1)} - \frac{1}{6\mu^2(N-1)^2} - \frac{1}{k} \right) + O\left(\frac{1}{N^4}\right) \right) \mod 2.
\end{aligned}$$

Multiplying both sides of the equation (23) by the scaling factor $\frac{N-1}{\alpha}$ and using the expansion

$$\frac{N-1}{(1 + [\mu(N-1)])} = \frac{1}{\mu} \left(\frac{1}{1 - \frac{1}{\mu(N-1)} - y} \right) = \frac{1}{\mu} \sum_{k=0}^{\infty} \left(1 - \frac{1}{\mu(N-1) + 1} \right)^{k+1} y^k$$

we obtain

$$\begin{aligned}
(24) \quad \frac{N-1}{\alpha} \varphi' &= \tilde{\varphi} + \left(\frac{2}{\mu} - 1 \right) - \frac{1}{6(N-1)} \left(\frac{1}{\mu^2} - 1 \right) - \frac{1}{\mu(1 + \mu(N-1))} + O\left(\frac{1}{N^3}\right) + \\
&+ \sum_{k=1}^{\infty} \left(\left(1 - \frac{1}{\mu(N-1) + 1} \right)^{k+1} + \frac{1}{\mu(N-1)} - \frac{1}{6\mu^2(N-1)^2} - \frac{1}{k} \right) y^k.
\end{aligned}$$

Since $\varphi \in \mathcal{I}_+$ we can use (19)

$$\tilde{\varphi}' = \frac{N}{\alpha} \varphi' = \left(1 + \frac{1}{N-1} \right) \frac{N-1}{\alpha} \varphi'.$$

Thus, in the leading order in the rescaled variables, the transformation takes the form

$$\tilde{\varphi}' = \tilde{\varphi} + \left(\frac{2}{\mu} - 1 \right).$$

REMARK 1. Note that in case $\varphi \in \mathcal{I}_0$ or $\varphi \in \mathcal{I}_-$, it is not necessarily true that $n = [\mu(N-1)] - N$, however the error of such approximation $e(N, \mu) = |n - [\mu(N-1)] - N|$ is bounded by a constant independent of N . Such small correction does not have an effect on the leading order asymptotics below.

Case $\varphi \in \mathcal{I}_0$. In this case $n = n'$ and so rescaling for the image φ' coincides with the rescaling for φ . On the other hand in this case $\Delta S = \frac{1}{N-1}$ so, for the rescaled variables we get from (24) : $\tilde{\varphi}' = \tilde{\varphi} + \frac{2}{\mu} + O(N^{-1})$.

Case $\varphi \in \mathcal{I}_-$. In the third region $n = n' - 1$ and so $\Delta S = \frac{1}{N-1} + \frac{1}{N-2}$. Rescaling gives $\tilde{\varphi}' = \frac{N-2}{\alpha} \varphi' = \left(1 - \frac{1}{N-1}\right) \frac{N-1}{\alpha} \varphi' + 2 + \frac{1}{N-2}$ which again gives from (24) $\tilde{\varphi}' = \tilde{\varphi} + \frac{2}{\mu} + O(N^{-1})$.

□

5.4. Proof of Theorem 2.

Proof. If $\alpha = \frac{1}{\ln n}$ then $\mu = n$ and therefore $y \equiv 0$. Thus, all the terms in series expansion (24) vanish. In the leading order, the rescaled transformation takes the form

$$\tilde{\varphi}' = \tilde{\varphi} + \left(\frac{2}{\mu} - 1\right),$$

i.e. the rigid rotation on the circle.

For our particular choice $\mu = 2$, in the leading order the transformation T is the identity map.¹ Thus, we get

$$\tilde{\varphi}' = \tilde{\varphi} + \frac{1}{8(N-1)} - \frac{1}{4N-2} + O(N^{-3})$$

To construct an unbounded orbit, we search for an initial point $\tilde{\varphi}_0$ satisfying two conditions:

- $\chi_+(\tilde{\varphi}_0) = 1$,
- The image $\tilde{\varphi}'_0$ under the transformation T coincides with initial angle up to the new normalization $\tilde{\tilde{\varphi}}_0$.

Note that for a larger action variable the scaling factor would be different. Therefore, we have

$$\tilde{\tilde{\varphi}}_0 = \frac{N-1}{N} \tilde{\varphi}_0 = \tilde{\varphi}_0 - \frac{1}{N} \tilde{\varphi}_0$$

and the “renormalized fixed point” condition yields

$$\tilde{\varphi}_0 - \frac{1}{N} \tilde{\varphi}_0 = \tilde{\varphi}_0 + \frac{1}{8(N-1)} - \frac{1}{4N-2} + O(N^{-3}).$$

Thus, up to the order $O(N^{-3})$

$$\begin{aligned} -\frac{\tilde{\varphi}_0}{N} &= \frac{4N-2-8N+8}{8(N-1)(4N-2)} = -\frac{2N-3}{8(N-1)(2N-1)} = \\ &= -\frac{1}{8(N-1)} + \frac{1}{4(N-1)(2N-1)} = -\frac{1}{8N} - \frac{1}{8N(N-1)} + \frac{1}{4(N-1)(2N-1)} = \\ &= -\frac{1}{8N} + \frac{1}{4(N-1)} \left(\frac{1}{2N} - \frac{1}{2N-1} \right) = -\frac{1}{8N} - \frac{1}{8N(N-1)(2N-1)}. \end{aligned}$$

As a result, for $\tilde{\varphi}_0 = \frac{1}{8} + O(N^{-3})$ its image under first return map is given by $\tilde{\varphi}'_0 = \frac{1}{8} + O(N^{-3})$.

¹For other values $\mu \in \mathbb{N}$, we would have periodic mappings.

To check the first statement $\chi_+(\varphi_0) = 1$ it is sufficient to show that $\varphi_0 + 2\alpha S_1 + \frac{\alpha}{N+n+1}$ is greater than 2. This immediately follows from (22) and the estimate

$$\varphi_0 + 2\alpha S_1 + \frac{\alpha}{N+n+1} > \frac{1}{8N \ln 2} + 2 - \frac{1}{\ln 2(2N-2)} + O(N^{-2}) + \frac{1}{(2N-1) \ln 2} > 2.$$

□

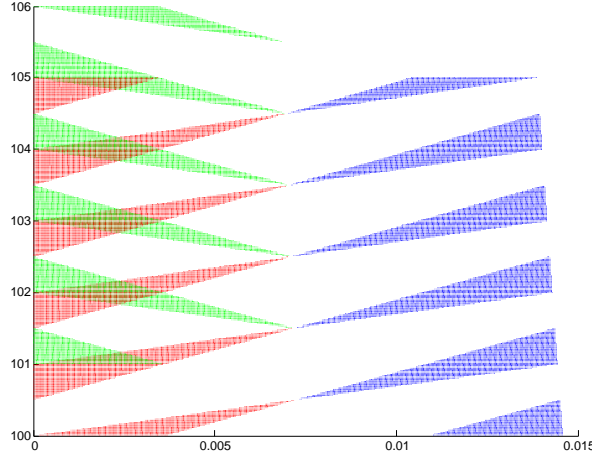


FIGURE 4. The structure of the fundamental domain for $\alpha = \frac{1}{\log 2}$. The regions of positive growth are in red and their images are in green. The regions of negative growth are in blue.

REMARK 2. Construction of the unbounded orbit in the Theorem 2 can be easily extended for any $\alpha = \frac{1}{\ln 2^m}$, $m \in \mathbb{N}$. Lemmas 1 and 2 do not depend on α and can be applied without any modifications. The only difference with the considered case $\mu = 2$ is in the action of T on the neutral set \mathcal{I}_0 . Instead of the pure rotation, there will be an additional interval exchange transformation acting on \mathcal{I}_0 with $[\mu] - 2$ subintervals. We omit technical details of this construction.

REMARK 3. We conjecture that the same arguments as in Lemmas 1 – 3 can be used for other values of $z \in (-1, 0)$. Here instead of asymptotic expansion for harmonic numbers one should use generalized harmonic numbers $H_{n,z}$. Since $H_{n,z} \rightarrow \zeta(k)$ as $n \rightarrow \infty$ one can establish the relation for $H_{(I+n),z} - H_{(I-1),z} = 2$ to find the expression for n . Arguments in Lemma 1, 2 and 3 can be applied with minor changes.

APPENDIX A.

In the appendix we give more detailed derivation of some specific problems that lead to αz map.

Example 1: Particle in switching potential. Consider a classical particle moving on the line in the square wave potential $V(x) = (-1)^{[x]}$ and assume the potential is switched every time unit $V(x, t) = (-1)^{[t]} \cdot (-1)^{[x]}$. While such potential is not differentiable, there is a natural way to define the dynamics by using the energy relation: the kinetic energy changes by 2 if the particle passes $t \in \mathbb{Z}$ integer points. We should ignore the singular subset of the extended phase space (t, \dot{x}, x) where there is discontinuity in both time and space and the dynamics is not defined. Such subset has zero measure. Outside the singular set particle moves with constant speed $v(x, t) = \sqrt{E - V(x, t)}$.

Since the dynamics defined for $x \bmod 2$, $t \bmod 2$ we can project the dynamics on a plane onto the system on a cylinder $[0, 2) \times \mathbb{R}_+$. Write the Hamiltonian form of the unit time step transformation for this system. Hamiltonian has a form $H(x, \dot{x}, t) = \frac{\dot{x}^2}{2} + V'(x, t)$ and so for the canonical action-angle variables we get

$$(25) \quad \begin{cases} I_{\pm} = \sqrt{2(H-1)} + \sqrt{2(H+1)} \\ \varphi_{\pm} = \frac{1}{T} \left(\frac{x-1}{\sqrt{2(H \pm \operatorname{sgn}(x-1))}} + \frac{1}{\sqrt{2(H \mp 1)}} \right) \end{cases}$$

Here $T = \left(\frac{1}{\sqrt{2(H-1)}} + \frac{1}{\sqrt{2(H+1)}} \right)$ is a period of rotation along the level curve $H(x, \dot{x}) = H$ and (φ_{\pm}, I_{\pm}) correspond to the angle-action variables for odd/even values of $[t]$. To deduce the system in action-angle variables, write down a generating function. For Hamiltonian we have

$$H(x, \dot{x}, t) = H_0(I, t) + \partial_3 S_{\pm}(x(t), I, t)$$

Where

$$\begin{cases} S_+(x, I) = (x-1)\sqrt{2(I + \operatorname{sgn}(x-1))} + \sqrt{2(I-1)} \\ S_-(I) = iS_+(-I) \end{cases}$$

Or, using (25)

$$S_+ = \begin{cases} 2T(I+1)\varphi_+ - \frac{4}{\sqrt{2(I-1)}}, & x > 1 \\ 2T(I-1)\varphi_+, & x < 1 \end{cases}$$

Finally, the total system in variables (φ, I) can be deduced from the expression (25) and the relation $\varphi_{\pm} = \partial_I S_{\pm}$.

$$(26) \quad \begin{cases} \varphi_1 = (\varphi + \alpha\sqrt{I}) \bmod 2 \\ I_1 = I + \operatorname{sgn}(\varphi_1 - 1) \end{cases}$$

where α is some constant. Clearly, this system can be considered as a particular example of transformation (2) with $z = \frac{1}{2}$.

Example 2: Fermi-Ulam acceleration. In Fermi-Ulam problem the particle bounces between two walls. Assume that one wall is at rest $x = 0$ and the other moves periodically $x = p(t)$, $p(t+1) = p(t) > 0$. There is a standard transformation “stopping” the wall, see e.g. [14] with

$$x = p(t)y, \quad \tau = \int_0^t \frac{ds}{p^2(s)} ds.$$

In the new variables, the equation takes the form

$$y'' + \ddot{p}p^3y = 0,$$

where \prime denotes the derivative wrt τ . Evaluating the one period map, under the assumption that p is piecewise linear, we obtain

$$(27) \quad \begin{cases} y_2 = (y_1 + y'_1) \bmod 1 \\ y'_2 = y'_1 + y_2 \operatorname{sgn}(y_2 - 1/2). \end{cases}$$

This mapping is not a particular case of αz -map but it corresponds to the linear growth of the action when $z = 1$. While showing unbounded growth is relatively easy in this case, more challenging problem is to estimate the relative measure of bounded solutions. This has been done in [2].

Example 3: Outer billiards with degenerate boundary. Consider the outer billiard system. Let γ be a smooth strictly convex curve on the plane. Take any point $x \in \mathbb{R}^2$ outside of γ and let $l(x)$ be a ray tangent to γ and oriented in the counter-clockwise direction. There is another point on the ray $T(x)$ which has the same distance to the tangency point as x . This defines the outer billiard map, see e.g. [13]. A natural question is whether all the orbits are bounded. Thus, one is led to study this map for large x .

Assume that γ is a unit circle centered at the origin, then for large x the square of the map T^2 is close to identity and it leaves concentric circles invariant. The angle changes by a factor of $1/|x|$, which corresponds to $z = -1$ in the αz -map. Now, consider the circle with a small circle segment removed. Then, the map becomes a small discontinuous perturbation of the above integrable map. When, γ is half the circle, the outer billiard has unbounded orbit, see [4, 3].

Earlier, Schwartz [12] constructed unbounded orbits for quadrilateral γ .

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